GENERALIZED GAUSSIAN EFFECTIVE POTENTIAL:

SECOND ORDER THERMAL CORRECTIONS

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ABSTRACT

We discuss the finite temperature generalized Gaussian effective potential. We put out a very simple relation between the thermal corrections to the generalized Gaussian effective potential and those of the effective potential. We evaluate explicitly the second order thermal corrections in the case of the selfinteracting scalar field in one spatial dimension.

In a previous paper [1] (henceforth referred to as I) we investigated the thermal corrections to the generalized Gaussian effective potential [2]. The starting point was the set-up of a perturbation theory with a variational basis which allowed to evaluate in a systematic way the corrections to the variational Gaussian approximation. In Ref. [2] we showed that the Hamiltonian H of a selfinteracting scalar field can be naturally decomposed into a free term H_0 and an interacting term H_I. It turns out that H_0 is the hamiltonian of a free scalar field with mass $\mu(\phi_0)$ satisfying the so called gap equation (in ν spatial dimensions)

$$\mu^2 = m^2 + \frac{\lambda}{2}\phi_0^2 + \frac{\lambda}{4} \int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{\sqrt{\vec{k}^2 + \mu^2}}. \tag{1}$$

The interaction H_I is given by the off-diagonal elements of the full Hamiltonian H with respect to the variational basis. Moreover, we showed that

$$H_I = \int d^\nu x \left[\left(\mu^2 \phi_0 - \frac{\lambda}{3}\phi_0^3 \right) : \eta(\vec{x}) : + \frac{\lambda}{3!}\phi_0 : \eta^3(\vec{x}) : + \frac{\lambda}{4!} : \eta^4(\vec{x}) : \right], \tag{2}$$

where $\eta(\vec{x})$ is the fluctuating field, and the colon denotes normal ordering with respect to the variational ground state.

In I we evaluated the thermal corrections to the generalized Gaussian effective potential V_G^T by means of the standard thermodinamic perturbation theory [3]. In this way we obtained

$$V_G^T(\phi_0) = V_{GEP}(\phi_0) + \frac{1}{\beta} \int \frac{d^\nu k}{(2\pi)^\nu} ln\left(1 - e^{-\beta g(\vec{k})}\right) + \Delta V_G^T(\phi_0) \tag{3}$$

where $g(\vec{k}) = \sqrt{\vec{k}^2 + \mu^2(\phi_0)}$ and (V is the spatial volume)

$$\Delta V_G^T(\phi_0) = -\frac{1}{\beta V} \sum_{m=2}^{\infty} \frac{(-1)^m}{m!} \int_0^\beta d\tau_1...\tau_m < T_\tau(H_I(\tau_1)...H_I(\tau_m) >_c^\beta . \tag{4}$$

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In Equation (4) $H_I(\tau)$ is the interaction Hamiltonian in the Matsubara interaction picture [4], and the thermal average is done with respect the free Hamiltonian H_0.

The aim of the present paper is to discuss some consequences of the general formulae (3) and (4). Moreover we evaluate the second order thermal corrections in the case of selfinteracting scalar field in one spatial dimension.

In the lowest order approximation, the thermal generalized Gaussian effective potential is given by Eq. (3) with $\Delta V_G^T(\phi_0) = 0$. It is interesting to compare our result with the finite temperature Gaussian effective potential discussed by Hajj and Stevenson [5] (see also Ref. [6]).

In order to evaluate the thermal corrections to the Gaussian effective potential within a non-perturbative approach, the authors of Ref. [5] decomposed the Hamiltonian into two terms:

$$H = H_0 + H_I, \tag{5}$$

where H_0 is the Hamiltonian of free scalar particle with variational mass M. Whereupon one evaluates the free energy by using the standard thermodinamic perturbation theory. Hajj and Stevenson, after evaluating the thermodinamic potentials up to the first order in the perturbation H_I, fix the variational mass M by minimizing the free energy density. As a consequence the mass M satisfies a gap equation which includes the thermal corrections. This is the main difference between our approach and the one of Ref. [5]. Indeed in our approach the variational basis is fixed once and for all at $T = 0$ by Eq. (1). We stress that in Ref. [5] the thermal variational mass and the interaction Hamiltonian depend on the approximation adopted in evaluating the free energy density. On the other hand, in our approach the interaction Hamiltonian is determined by the variational basis at T=0.

It is worthwile to compare our lowest order thermal corrections with the 1-loop finite temperature effective potential [7,8]. In the one loop approximation the

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finite temperature effective potential is well known:

$$V_\beta^1(\phi_0) = \frac{1}{2\beta} \sum_n \int \frac{d^\nu k}{(2\pi)^\nu} ln(E^2 + \omega_n^2) \tag{6}$$

where $\omega_n = \frac{2\pi n}{\beta}$, $E^2 = \vec{k}^2 + M^2(\phi_0)$, and $M^2(\phi_0) = m^2 + \frac{\lambda}{2}\phi_0^2$. The sum over n can be readily evaluated [7]:

$$V_\beta^{(1)}(\phi) = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{E}{2} + \frac{1}{\beta} \int \frac{d^\nu k}{(2\pi)^\nu} ln(1 - e^{-\beta E}). \tag{7}$$

The first term in Eq. (7) is the zero temperature 1-loop effective potential. Thus the 1-loop thermal correction is given by the second term in (7). Now, comparing Eq. (7) with Eq. (3) we see that 1-loop thermal correction to the effective potential coincides with the lowest order thermal correction to the generalized Gaussian effective potential if

$$M^2(\phi_0) = m^2 + \frac{\lambda}{2}\phi_0^2 \longrightarrow \mu^2(\phi_0). \tag{8}$$

In other words, if in the thermal correction to the effective potential we replace the tree level mass of the shifted theory with the mass $\mu(\phi_0)$ obtained by summing the superdaisy graphs at $T = 0$ in the propagator, then we obtain again a free energy density. Up to now this remarkable result in thermal scalar field theories holds for the lowest order correction only. We show, now, that it extends also to the higher order thermal corrections. To this end we observe that the higher order corrections are given by Eq. (1.9) of Ref. [7]. On the other hand, in our approach the thermal corrections can be evaluated by means of Eq. (4). Observing that $L_I = -H_I$ and that the Gaussian functional integration with periodic boundary conditions in Ref. [7] corresponds to the thermal Wick theorem, we obtain the desired result. However, because our interaction Hamiltonian is normal ordered, to complete the proof we must show that the thermal corrections are not affected

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by the normal ordering of the interaction Hamiltonian. To see this, we note that the normal ordering is ineffective when we consider a thermal contraction of of two scalar fields belonging to different vertices. Therefore, the normal ordering comes into play when we contract two fields which belong to the same vertex. In this case we get the following thermal average

$$\tilde{G}_\beta(0) = < T_\tau : \eta(\vec{x}, \tau)\eta(\vec{x}, \tau) :>^\beta, \tag{9}$$

instead of $G_\beta(0)$, where $G_\beta(\vec{x}, \tau)$ is the thermal propagator:

$$G_\beta(\vec{x}, \tau) = < T_\tau \eta(\vec{x}, \tau)\eta(0) >^\beta = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^\nu k}{(2\pi)^\nu} \frac{e^{i[\vec{k}\cdot\vec{x} - \omega_n \tau]}}{\omega_n^2 + g^2(\vec{k})}. \tag{10}$$

Taking into account the canonical commutation relations between the creation and annihilation operators, it is straighforward to show that:

$$\tilde{G}_\beta(0) = G_\beta(0) - \int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{2g(\vec{k})}. \tag{11}$$

Now we observe that

$$\int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{2g(\vec{k})} = \lim_{\beta \to \infty} G_\beta(0). \tag{12}$$

Indeed, from Eq. (10) it follows:

$$G_\beta(0) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{\omega_n^2 + g^2(\vec{k})}. \tag{13}$$

By using the identity

$$\text{cotgh}(x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}, \tag{14}$$

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we rewrite Eq. (13) as:

$$G_\beta(0) = \int \frac{d^\nu p}{(2\pi)^\nu} \frac{1}{2g(\vec{p})} \text{cotgh}\left[\frac{\beta g(\vec{p})}{2}\right]. \tag{15}$$

Finally, performing the limit $\beta \to \infty$ in Eq. (15) we obtain Eq. (12). Thus, we have shown that the normal ordering of the interaction Hamiltonian does not modify the thermal corrections.

Note that from Eqs. (11) and (15) it follows that

$$\tilde{G}_\beta(0) = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{2g(\vec{k})} \left[\text{cotgh}\left(\frac{\beta}{2}g(\vec{k})\right) - 1\right]. \tag{16}$$

Equation (16) shows that $\tilde{G}_\beta(0)$ is finite for any value of ν.

Let us, now, evaluate the second order thermal corrections in the case of one spatial dimension, $\nu = 1$. From Eq. (4) we have

$$\Delta V_G^T(\phi_0) = -\frac{1}{2!\,\beta V} \int_0^\beta d\tau_1 d\tau_2 < T_\tau H_I(\tau_1) H_I(\tau_2) >_c^\beta. \tag{17}$$

In Figure 1 we display the second oder thermal corrections obtained with the aid of the thermal Wick theorem. The solid lines correspond to the thermal propagator Eq. (10), the vertices can be extracted from the interaction Hamiltonian Eq. (2). Let us analyze the graphs in Fig. 1. It is easy to see that graph (a) is temperature-independent. So it does not contribute to ΔV_G^T due to the stability condition $< \Omega\,|\,\eta\,|\,\Omega >= 0$. As concern the graph (b), we have

$$(b) = -\frac{\lambda\phi_0}{4\beta V}\left(\mu^2\phi_0 - \frac{\lambda}{3}\phi_0^3\right) \int_{-\infty}^{+\infty} dx\,dy \int_0^\beta d\tau_1\,d\tau_2 < T_\tau \eta(x,\tau_1)\eta(y,\tau_2) >^\beta$$
$$< T_\tau : \eta(y,\tau_2)\eta(y,\tau_2) :>^\beta.$$

According to our previous discussion we obtain

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$$(b) = -\frac{1}{4}\lambda\phi_0^2\left(1 - \frac{\lambda}{3\mu^2}\phi_0^2\right)\tilde{G}_\beta(0). \tag{18}$$

In a similar way we find:

$$(c) = -\frac{\lambda^2\phi_0^2}{8}\frac{1}{\mu^2}\tilde{G}_\beta^2(0). \tag{19}$$

For the graphs (d) we have

$$(d) = -\frac{\lambda^2\phi_0^2}{2\cdot 3!}\int_{-\frac{\beta}{2}}^{+\frac{\beta}{2}} d\tau \int_{-\infty}^{+\infty} dx\, G_\beta^3(x,\tau). $$

Using Eq. (10) and the result

$$\frac{1}{\beta}\sum_n \frac{e^{i\omega_n\tau}}{\omega_n^2 + g^2(k)} = \frac{e^{-g(k)|\tau|}}{2g(k)} + \frac{1}{2g(k)}\frac{(e^{g(k)\tau} + e^{-g(k)\tau})}{e^{\beta g(k)} - 1}, \tag{20}$$

we get

$$(d) = -\frac{\lambda^2\phi_0^2}{48(2\pi)^2}\int_0^{\frac{\beta}{2}} d\tau \int_{-\infty}^{+\infty} \frac{dk_1\, dk_2}{g(k_1)g(k_2)g(k_3)}\prod_{i=1}^3\left[e^{-g(k_i)\tau} + \frac{e^{g(k_i)\tau} + e^{-g(k_i)\tau}}{e^{\beta g(k_i)} - 1}\right] \tag{21}$$

where $\sum_{i=1}^3 k_i = 0$.

Finally, using Eq. (20) we get:

$$(e) = -\frac{\lambda^2}{32(2\pi)}\tilde{G}_\beta^2(0)\int_0^{\frac{\beta}{2}} d\tau \int_{-\infty}^{+\infty} \frac{dk}{g^2(k)}\left[e^{-g(k)\tau} + \frac{e^{g(k)\tau} + e^{-g(k)\tau}}{e^{\beta g(k)} - 1}\right]^2 \tag{22}$$

and

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$$(f) = -\frac{\lambda^2}{16 \cdot 4!(2\pi)^3} \int_0^{\frac{\beta}{2}} d\tau \int_{-\infty}^{+\infty} \frac{dk_1 dk_2 dk_3}{g(k_1)g(k_2)g(k_3)g(k_4)} \cdot$$
$$\prod_{i=1}^4 \left[e^{-g(k_i)\tau} + \frac{e^{g(k_i)\tau} + e^{-g(k_i)\tau}}{e^{\beta g(k_i)} - 1} \right] \qquad (23)$$

with $\sum_{i=1}^4 k_i = 0$.

A few comments are in order. In Equations (21), (22) and (23) the τ-integration can be performed explicitly, while the remaining integrations over the momenta k_i must be handled numerically. In the limit $\beta \to \infty \, (T \to 0)$ the anomalous "graphs" (b), (c) and (e) go to zero exponentially due to the factor $\tilde{G}_\beta(0)$. On the other hand, the graphs (d) and (f) reduce to the zero temperature second order corrections to the Gaussian effective potential [9]. Indeed, in that limit in Eqs. (21-23) only the factor $e^{-g(k_i)\tau}$ survives. Performing the elementary τ-integration we obtain the zero temperature contributions. As a consequence the zero temperature limit of $V_G^T(\phi_0)$ reduces to $V_G(\phi_0)$.

In the high temperature limit $\beta \to 0$ we find that the graphs (e) and (f) dominate. Therefore, in the intermediate temperature region $\beta \sim 1$ we expect that the main contribute to $V_G^T(\phi_0)$ comes from the graphs (d), (e) and (f). Indeed this is the case as shown in Fig. 2.

In Figure 3 we display the finite temperature generalized Gaussian effective potential for three different values of T and $\hat{\lambda} > \hat{\lambda}_c \simeq 1.15$ [9]. Figure 3 shows that the symmetry broken at $T = 0$ is restored by increasing the temperature through a continuous phase transition.

In conclusion, in this paper we have discussed the thermal corrections to the generalized Gaussian effective potential. Remarkably, we found that the thermal corrections can be obtained from those of the effective potential with the substitution Eq. (8). Moreover, we have evaluated the second order corrections in the case of selfinteracting scalar fields in one spatial dimension. We plan to extend our work to the case of higher spatial dimensions in a future investigation.

FIGURE CAPTIONS

Figure 1 Second order thermal corrections to the generalized Gaussian effective potential.

Figure 2 Contributions to $V_G^T(\phi_0)$ due to the graphs in Fig. 1 for $\hat{\lambda} = 4$ and $\hat{T} = 1$ (notations as in I).
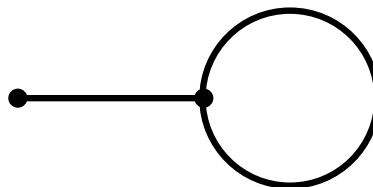
Figure 3 $V_G^T(\phi_0)$ in units of $\mu_0 = \mu(\phi_0 = 0)$ for $\hat{\lambda} = 4$ and $\hat{T} = 0$, $\hat{T} = \hat{T}_c \simeq 0.764$, and $\hat{T} > \hat{T}_c$.
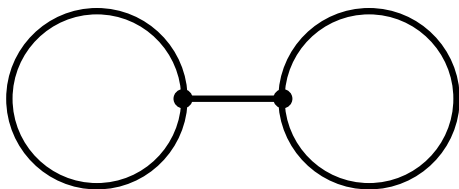
REFERENCES

[1] P. Cea and L. Tedesco, *Finite Temperature Generalized Gaussian Effective Potential*, BARI-TH/198/95.

[2] P. Cea, *Phys. Lett.* **B 236** (1990) 191;
P. Cea and L. Tedesco, *Phys. Lett.* **B 335** (1994) 423.

[3] See, for istance, A.A. Abrikosov, L.P. Gorkov, and I.E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1975);
A.L. Fetter and J.D.Walecka, *Quantum Theory of Many Particle System* (Mc Graw-Hill, New York, 1965).

[4] T. Matsubara, *Prog. Theor. Phys.* **177** (1955) 351.

[5] G.A. Hajj and P.M. Stevenson, *Phys. Rev.* **D 37** (1988) 413.

[6] I. Roditi, *Phys. Lett.* **B 177** (1986) 85;
A. Okopinska, *Phys. Rev.* **D 36** (1987) 2415.

[7] L. Dolan and R. Jackiw, *Phys. Rev.* **D 9** (1974) 3320.

[8] S. Weinberg, *Phys. Rev.* **D 9** (1974) 3357.

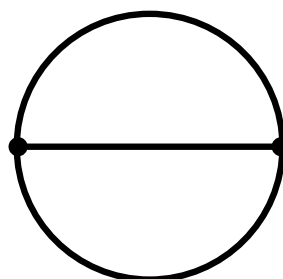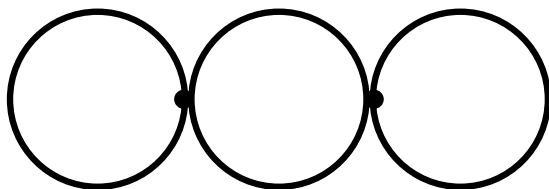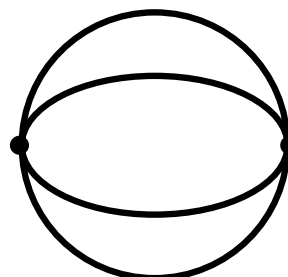[9] L. Tedesco, *Ph. D. Thesis*, University of Bari, 1995 (unpublished).

(a)

(b)

(c)

(d)

(e)

(f)

FIGURE 1

FIGURE 2

FIGURE 3